Chapter 6

Recap Sheet

- **Difference of two means with both variances known (two sample z inference)**

What question? Compare the average of population 1 and population 2. Parameter of interest : $\Delta = \mu_1 - \mu_2$

Do the data suggests Δ equals to some value (often 0)? And what's a CI for Δ?

Assumptions :

1. Two independant random samples :
   $X_{1,1}, \dots, X_{1,n_1}$ for population 1
   $X_{2,1}, \dots, X_{2,n_2}$ for population 2
2. Each population is Normal (or samples large enough for CLT)
3. Population standard deviations σ_1 and σ_2 are known.

Sampling result you use : The difference of sample means

$$\overline{X}_1 - \overline{X}_2 \sim \mathcal{N}\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

Test statistic (z) : For testing $H_0 : \mu_1 - \mu_2 = \Delta_0$ (often $\Delta_0 = 0$)

$$Z_0 = \frac{(\overline{X}_1 - \overline{X}_2) - \Delta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \overset{H_0}{\sim} \mathcal{N}(0,1)$$

Decision rules (level α) :

- Two-sided $H_1 : \mu_1 - \mu_2 \neq \Delta_0$: reject if $|Z_0| \geq z_{1-\frac{\alpha}{2}}$
- Upper-tailed $H_1 : \mu_1 - \mu_2 > \Delta_0$: reject if $Z_0 \geq z_{1-\alpha}$
- Lower-tailed $H_1 : \mu_1 - \mu_2 < \Delta_0$: reject if $Z_0 \leq z_\alpha$

Confidence interval for Δ :

$$(\overline{X}_1 - \overline{X}_2) \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Ex : Two formulations, known $\sigma_1 = \sigma_2 = 8$ min, $n_1 = n_2 = 10$
Sample means : $\bar{x}_1 = 121$, $\bar{x}_2 = 112$.
Test $H_0 : \mu_1 - \mu_2 = 0$ vs $H_1 : \mu_1 > \mu_2$ at $\alpha = 0.05$.

$SE = \sqrt{\frac{8^2}{10} + \frac{8^2}{10}} = 3.577$. $Z_0 = \frac{121 - 112}{3.577} = 2.52$

Upper-tailed : compare to $z_{1-0.05} = 1.645 < 2.52 \rightarrow$ reject H_0
Formulation 1 has a larger mean drying time.
CI (95%) = $9 \pm 1.96 \times 3.577 = [1.99, 16.01]$ minutes

- Difference of two means with unknown but equal variances (pooled two-sample t inference)

Assumptions :
- Two independant random samples from Normal populations
  - Sample 1 : $X_{1,1}, \dots, X_{1,n_1}$ from $\mathcal{N}(\mu_1, \sigma^2)$
  - Sample 2 : $X_{2,1}, \dots, X_{2,n_2}$ from $\mathcal{N}(\mu_2, \sigma^2)$
- The population variances are equal : $\sigma_1^2 = \sigma_2^2 = \sigma^2$ (unknown)

Pooled variance (combine both sample variances) :
Let s_1^2, s_2^2 be the sample variances

$$S_p^2 = \frac{(n_1 - 1)\, s_1^2 + (n_2 - 1)\, s_2^2}{n_1 + n_2 - 2}$$

Degrees of freedom : $\nu = n_1 + n_2 - 2$

Test statistic (two-sample pooled t) :
To test $H_0 : \mu_1 - \mu_2 = \Delta_0$ (usually $\Delta_0 = 0$) :

$$T_0 = \frac{(\overline{X}_1 - \overline{X}_2) - \Delta_0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \overset{H_0}{\sim} t_\nu$$

Decision rules (level α) :
- Two-sided H_1 : $\mu_1 - \mu_2 \neq \Delta_0$: reject if $|T_0| \geq t_{1-\frac{\alpha}{2},\nu}$
- Upper-tailed H_1 : $\mu_1 - \mu_2 > \Delta_0$: reject if $T_0 \geq t_{1-\alpha,\nu}$
- Lower-tailed H_1 : $\mu_1 - \mu_2 < \Delta_0$: reject if $T_0 \leq t_{\alpha,\nu}$

Confidence interval for $\mu_1 - \mu_2$:

$$(\overline{X}_1 - \overline{X}_2) \pm t_{1-\frac{\alpha}{2},\nu} \underbrace{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}_{SE_p \text{ (std error)}}$$

Ex : Two catalysts are compared. Assume equal variances.
Data summaries : $\bar{x}_1 = 92.255$, $s_1 = 0.35$, $n_1 = 8$
$\bar{x}_2 = 92.733$, $s_2 = 2.98$, $n_2 = 8$
Test $H_0 : \mu_1 - \mu_2 = 0$ vs $H_1 : \mu_1 \neq \mu_2$ at $\alpha = 0.05$.

1. $S_p^2 = \frac{7 \times 0.35^2 + 7 \times 2.98^2}{14} = 4.513 \rightarrow S_p = 2.125$
2. $SE_p = 2.125 \sqrt{\frac{1}{8} + \frac{1}{8}} = 1.0625$
3. $T_0 = \frac{92.255 - 92.733}{1.0625} = -0.45$
4. $\nu = 14$; $t_{0.975,14} \approx 2.145$. Since $|T_0| = 0.45 < 2.125 \rightarrow$ fail to reject H_0.

(95% CI $\approx -0.478 \pm 2.145 \times 1.0625 \Rightarrow [-2.76 ; 1.80]$, includes 0)

- Difference of two means with unknown and unequal variances (Welsh two-sample t)

This is the safe default when you cannot assume equal variances.

Assumptions :

- Two independant random samples from Normal populations
  - Sample 1 : size n_1, mean $\overline{x}_1$, variance s_1^2
  - Sample 2 : size n_2, mean $\overline{x}_2$, variance s_2^2
- Variances are unknown and not assumed equal : $\sigma_1^2 \neq \sigma_2^2$

Standard error and test statistic : Estimate $\Delta = \mu_1 - \mu_2$.
For testing $H_0 : \Delta = \Delta_0$ (often 0) :

$$SE_w = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}, \quad T_0 = \frac{(\overline{x}_1 - \overline{x}_2) - \Delta_0}{SE_w}$$

Reference distribution : approximately t_ν, with Welsh-Satterthwaite df

$$\nu = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}}$$ (round to nearest integer)

Decision rules (level α) :

- Two-sided $H_1 : \mu_1 - \mu_2 \neq \Delta_0$: reject if $|T_0| \geq t_{1-\frac{\alpha}{2},\nu}$
- Upper-tailed $H_1 : \mu_1 - \mu_2 > \Delta_0$: reject if $T_0 \geq t_{1-\alpha,\nu}$
- Lower-tailed $H_1 : \mu_1 - \mu_2 < \Delta_0$: reject if $T_0 \leq t_{\alpha,\nu}$

Confidence interval for $\mu_1 - \mu_2$:

$$(\overline{x}_1 - \overline{x}_2) \pm t_{1-\frac{\alpha}{2},\nu} SE_w$$

Ex : Given $\overline{x}_1 = 92.255$, $s_1 = 0.35$, $n_1 = 8$
$\overline{x}_2 = 92.733$, $s_2 = 2.98$, $n_2 = 8$

Test $H_0 : \mu_1 - \mu_2 = 0$ vs $H_1 : \mu_1 \neq \mu_2$ at $\alpha = 0.05$

1. $SE_w = \sqrt{\frac{0.35^2}{8} + \frac{2.98^2}{8}} = 1.0609$
2. $T_0 = \frac{92.255 - 92.733}{1.0609} = -0.451$
3. $\nu = \frac{(0.01531 + 1.11005)^2}{\frac{0.01531^2}{7} + \frac{1.11005^2}{7}} = 7.20 \Rightarrow \nu = 7$
4. Critical value $t_{.975,7} = 2.365$. Since $|T_0| = 0.451 < 2.365$, fail to reject H_0.
5. 95% CI : $(-0.478) \pm 2.365(1.0609) \Rightarrow [-2.99, 2.03]$ which includes 0

- **Comparing two variances with the F test**

Goal : Decide whether the population variances are equal :

$H_0 : \sigma_1^2 = \sigma_2^2$ vs $H_1 : \sigma_1^2 \neq \sigma_2^2$ (or one-sided $\sigma_1^2 > \sigma_2^2$ or $\sigma_1^2 < \sigma_2^2$)

Assumptions : Two independant random samples from Normal pops.
- Sample 1 : size n_1, sample variance s_1^2
- Sample 2 : size n_2, sample variance s_2^2

F distribution : If the data are Normal, then

$$\frac{(n_1 - 1) S_1^2}{\sigma_1^2} \sim \chi^2_{n_1 - 1} , \quad \frac{(n_2 - 1) S_2^2}{\sigma_2^2} \sim \chi^2_{n_2 - 1}$$

independant. Their ratio (each divided by its df) is F :

$$F_0 = \frac{S_1^2}{S_2^2} \sim F_{\nu_1, \nu_2} \text{ under } H_0 , \ \nu_1 = n_1 - 1 , \ \nu_2 = n_2 - 1$$

Decision rules (level α) :

Two-sided $H_1 : \sigma_1^2 \neq \sigma_2^2$

- If you don't reorder variances, reject H_0 if

$$F_0 \leq F_{\frac{\alpha}{2}, \nu_1, \nu_2} \quad \text{or} \quad F_0 \geq F_{1-\frac{\alpha}{2}, \nu_1, \nu_2}$$

- If you reorder to make $s_1^2 \geq s_2^2$ (so $F_0 \geq 1$), reject if

$$F_0 \geq F_{1-\frac{\alpha}{2}, \nu_1, \nu_2} \quad \left(\text{lower bound is } \frac{1}{F_{1-\frac{\alpha}{2}, \nu_2, \nu_1}}\right)$$

One-sided tests
- $H_1 : \sigma_1^2 > \sigma_2^2$, reject if $F_0 \geq F_{1-\alpha, \nu_1, \nu_2}$
- $H_1 : \sigma_1^2 < \sigma_2^2$, reject if $F_0 \leq F_{\alpha, \nu_1, \nu_2}$

Confidence interval for $\rho = \frac{\sigma_1^2}{\sigma_2^2}$:

$$\left[\frac{S_1^2}{S_2^2} \cdot \frac{1}{F_{1-\frac{\alpha}{2}, \nu_1, \nu_2}} , \ \frac{S_1^2}{S_2^2} \cdot \frac{1}{F_{\frac{\alpha}{2}, \nu_1, \nu_2}} \right]$$

Ex : Two etching gases, Normality is plausible.
$n_1 = n_2 = 16 \Rightarrow \nu_1 = \nu_2 = 15$
Sample SDs : $s_1 = 1.96 \Rightarrow s_1^2 = 3.84$, $s_2 = 2.13 \Rightarrow s_2^2 = 4.54$
Test $H_0 : \sigma_1^2 = \sigma_2^2$ vs $H_1 : \sigma_1^2 \neq \sigma_2^2$ at $\alpha = 0.05$.

1. Put the larger variance on top : $F_0 = \frac{4.54}{3.84} = 1.18$

2. Critical value : $F_{0.975, 15, 15} \approx 2.86$

3. Decision : $1.18 < 2.86 \longrightarrow$ fail to reject H_0.
   There isn't strong evidence that the variance differ.

• Inference on two population proportions $p_1 - p_2$

What question? Compare success rates in two independant groups.
Parameter $\Delta = p_1 - p_2$

Assumptions:

- Two independant random samples.
  - Group 1 : size n_1, success $X_1 \Rightarrow \hat{p}_1 = \frac{X_1}{n_1}$
  - Group 2 : size n_2, success $X_2 \Rightarrow \hat{p}_2 = \frac{X_2}{n_2}$
- Large-sample condition: $n_i \hat{p}_i \gtrsim 10$ and $n_i(1-\hat{p}_i) \gtrsim 10$

Hypothesis test of equality $H_0 : p_1 = p_2$. When H_0 is true, both groups share a common p Estimate it with the pooled proportion

$$\hat{p} = \frac{X_1 + X_2}{n_1 + n_2}$$

Test statistic (z):

$$Z_0 = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \overset{H_0}{\sim} \mathcal{N}(0,1)$$

Decision (level α):

- Two-sided $H_1 : p_1 \neq p_2$: reject if $|Z_0| \geqslant z_{1-\frac{\alpha}{2}}$
- Upper-tailed $H_1 : p_1 > p_2$: reject if $Z_0 \geqslant z_{1-\alpha}$
- Lower-tailed $H_1 : p_1 < p_2$: reject if $Z_0 \leqslant z_\alpha$

Confidence interval for $\Delta = p_1 - p_2$: For estimation you don't pool; use each group's own variability:

$$(\hat{p}_1 - \hat{p}_2) \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

Ex : Two groups of $n_1 = n_2 = 100$ patients:

- Treatment successes $X_1 = 27 \Rightarrow \hat{p}_1 = 0.27$
- Placebo successes $X_2 = 19 \Rightarrow \hat{p}_2 = 0.19$

Test $H_0 : p_1 = p_2$ vs $H_1 : p_1 \neq p_2$, $\alpha = 0.05$

1. Pooled $\hat{p} = \frac{27+19}{100+100} = 0.23$
2. SE under H_0 . $\sqrt{0.23 \times 0.77 \left(\frac{1}{100} + \frac{1}{100}\right)} = 0.1882$
3. $Z_0 = \frac{0.27 - 0.19}{0.1882} = 1.34$ 4. $z_{0.975} = 1.96$
5. $|Z_0| = 1.34 < 1.96 \rightarrow$ fail to reject H_0.

95% confidence interval: $SE = \sqrt{\frac{0.27 \times 0.73}{100} + \frac{0.19 \times 0.81}{100}} = 0.0589$

$(\hat{p}_1 - \hat{p}_2) \pm z_{1-\frac{\alpha}{2}} SE \Rightarrow [-0.035, 0.195] \rightarrow 0$ is inside the CI